J. Fluid Mech. (1976), *vol.* 75, *part* 4, *pp.* 777–790
Printed in Great Britain

On the Oseen linearization of the swirling-flow boundary layer

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(Received 2 October 1975)

The modified Oseen linearization of the swirling-flow boundary layer of a mature hurricane is discussed. Upper and lower solutions for the radial and azimuthal velocity components are constructed in a subdomain by using a comparison theorem. It is shown that the error incurred in these velocity components by using the linear solution is no more than 30 % in the subdomain. It is then inferred that the vertical velocity is also approximated to that order.

1. Introduction

In 1971, Carrier, Hammond & George studied a model of the mature hurricane. In the same year, Carrier (1971) studied an Oseen-like linearization of the boundary-layer equations which arise in swirling flows over a rigid boundary in a rotating container. The purpose of that study was to construct simple, yet reasonably accurate, approximate solutions to the set of nonlinear parabolic equations.

This method of linearization has often been used in fluid-mechanical problems and is applicable to a much wider class of problems than the swirling boundary layer. It is therefore desirable to consider whether an error estimate can be obtained in a rigorous manner, in so far as conveniently possible. The purpose of this study is to construct such an estimate. Here also it is the procedure that is of primary interest.

We first re-examine the linearized system to obtain a solution slightly different from that given in Carrier (1971). We then use a comparison theorem due to Nagumo (1939) and Westphal (1949) to obtain an error estimate. It is seen that the linearization procedure is quite accurate (the precise meaning of this is given in §7), and the 30 % error suggested by Carrier appears well within reason.

In §2, we formulate the problem and its linearization. In §3, we discuss the nature of the linearized solution. In §4, we discuss how the constants used in the linearization can be determined. In §5, we state the Nagumo lemma and the necessary definitions. In §§6 and 7, we construct upper and lower solutions for the radial and azimuthal velocity components by using the comparison theorem. We conclude by inferring that the vertical velocity is approximated by the linear solution to within 30 %.

2. Formulation

Let (r, θ, z) be cylindrical co-ordinates and (u, v, w) be the velocity components in the directions of increasing (r, θ, z). The system of equations to be considered is (Carrier *et al.* 1971)

$$uu_r + wu_z - v^2/r - 2\Omega v + V^2/r + 2\Omega V = \nu u_{zz},$$

$$u(rv)_r + w(rv)_z + 2\Omega(ru) = \nu(rv)_{zz},$$

$$rw_z + (ru)_r = 0,$$

$$0 < r_1 < r < r_0, \quad 0 < z < \infty.$$

Here, Ω and ν are positive constants, and $V = V(r)$ is a given function with the property that $V > 0$ and $V_r < 0$ in (r_1, r_0).

The boundary conditions to be imposed on (u, v, w) are that they vanish on $z = 0$, coincide with the Ekman solution as $r \to r_0$ and satisfy $u = 0$ and $v = V(r)$ as z tends to infinity.

We introduce non-dimensional quantities as follows:

$$V' = rV/\Psi_0, \quad v' = rv/\Psi_0, \quad u' = ru/\Psi_0,$$

$$z' = z(2\Omega/\nu)^{\frac{1}{2}}, \quad w' = w/(2\nu\Omega)^{\frac{1}{2}}, \quad x = r^2\Omega/\Psi_0,$$

where Ψ_0 is a dimensional quantity which characterizes the strength of the hurricane. In terms of the non-dimensional quantities, and dropping the primes, we have

$$uu_x + wu_z + (V^2 - v^2 - u^2)/2x + (V - v) = u_{zz}, \tag{1}$$

$$uv_x + wv_z + u = v_{zz}, \tag{2}$$

$$u_x + w_z = 0, \tag{3}$$

$$0 < x_1 < x < x_0, \quad 0 < z < \infty.$$

The boundary conditions are

$$\left.\begin{aligned} &u(x,0) = 0, \quad v(x,0) = 0,\\ &u(x_0,z) = -V(x_0)\,e^{-z/\sqrt{2}}\sin(z/\sqrt{2}),\\ &v(x_0,z) = V(x_0)\,[1 - e^{-z/\sqrt{2}}\cos(z/\sqrt{2})],\\ &u(x,\infty) = 0, \quad v(x,\infty) = V(x). \end{aligned}\right\} \tag{4}$$

We observe that the conditions at x_0 are chosen such that u and v merge with the Ekman solution for x close to x_0. It is assumed that the system (1)–(4) has a unique solution.

We shall not reproduce the arguments used by Carrier (1971) to arrive at the modified Oseen linearization of (1) and (2). In non-dimensional form, the equations are

$$\bar{u}_{zz} = \frac{CV}{x}\bar{u} - \left(1 + \frac{DV}{2x}\right)(\bar{v} - V), \quad C > 0, \quad D > 0, \tag{5}$$

$$\bar{v}_{zz} = \bar{u}, \tag{6}$$

and $\bar{u}$ and $\bar{v}$ are required to satisfy the boundary conditions (4). The positive constants C and D are to be determined *a posteriori*. In Carrier (1971), C and D were chosen from some global conservation criterion to be $\frac{8}{5}$ and $\frac{3}{4}$ respectively. Since in the subsequent error estimates we need the explicit solution of (5) and (6), and its region of validity, we shall first examine the solutions of these equations.

3. The modified Oseen solution

We combine (5) and (6) to get

$$\bar{u}_{zzzz} - \frac{CV}{x}\bar{u}_{zz} + \left(1 + \frac{DV}{2x}\right)\bar{u} = 0. \tag{7}$$

We seek solutions of the form $e^{\lambda z}$. In a straightforward manner, we can calculate λ as the roots of the algebraic equation

$$\Lambda^4 - \frac{CV}{x}\Lambda^2 + \left(1 + \frac{DV}{2x}\right) = 0. \tag{8}$$

We have
$$\Lambda_{1,2}^2 = \frac{CV}{2x} \pm \left[\left(\frac{CV}{2x}\right)^2 - \left(1 + \frac{DV}{2x}\right)\right]^{\frac{1}{2}}.$$

The solutions of (7) can be classified into three types, depending on the sign of the discriminant. Since the discriminant may change sign for x in the range (x_1, x_0), it appears that all three types may be admissible.

(i) If $(CV/2x)^2 - (1 + DV/2x) = 0$, then Λ^2 has a double root. Hence the roots of (8) are real and of opposite sign.

(ii) If $(CV/2x)^2 - (1 + DV/2x) > 0$, then Λ_1^2 and Λ_2^2 are real and positive, so that (8) has two real positive roots and two real negative ones.

(iii) If $(CV/2x)^2 - (1 + DV/2x) < 0$, then Λ_1^2 and Λ_2^2 are complex conjugates, and all roots of (8) are complex, two having positive real parts and the remaining two having negative real parts. In fact, let

$$\alpha = CV/2x, \quad \beta = [(1 + DV/2x) - \alpha^2]^{\frac{1}{2}},$$

so that
$$\Lambda_{1,2}^2 = \alpha \pm i\beta = (\alpha^2 + \beta^2)^{\frac{1}{2}} e^{\pm i\theta},$$

where $\theta = \tan^{-1}\beta/\alpha$, $0 < \theta < \frac{1}{2}\pi$; then the roots with negative real parts are

$$-(\alpha^2 + \beta^2)^{\frac{1}{4}} (\cos \tfrac{1}{2}\theta \pm i \sin \tfrac{1}{2}\theta).$$

In view of the boundary conditions to be imposed at x_0, it is clear that the type (iii) solution is appropriate, which requires $\beta^2 > 0$ at least for x close to x_0. The condition at (x_0, z) can be *approximately satisfied* if C and D are of order unity and $V(x_0)/x_0 \ll 1$, for then we have $\Lambda_{1,2}^2(x_0) \doteq \pm i$. The type (iii) solution then takes the form

$$\bar{u} = -V(\alpha^2 + \beta^2)\beta^{-1} \exp(-\lambda_1 z) \sin \lambda_2 z \tag{9}$$

and
$$\bar{v} = V - V \exp(-\lambda_1 z) \cos \lambda_2 z - V \cot\theta \exp(-\lambda_1 z) \sin \lambda_2 z, \tag{10}$$

where $\lambda_1(x) = (\alpha^2 + \beta^2)^{\frac{1}{4}} \cos \frac{1}{2}\theta$ and $\lambda_2(x) = (\alpha^2 + \beta^2)^{\frac{1}{4}} \sin \frac{1}{2}\theta$. We observe that, if

$\beta^2 > 0$ ($\theta > 0$) for x in (x_1, x_0), then we have the type (iii) solution in the entire interval of interest. If β^2 is positive for $\tilde{x} < x_0$ but vanishes ($\theta = 0$) at $\tilde{x}$, the type (iii) solution holds for $\tilde{x} < x < x_0$. Since λ_2 vanishes with β, an application of L'Hospital's rule shows that

$$\lim_{x \to \tilde{x}} \bar{u} = \tfrac{1}{2}\alpha^{\frac{3}{2}} z \exp(-\alpha^{\frac{1}{2}} z).$$

For $x_1 < x < \tilde{x}$, it is clear that the type (iii) solution no longer applies. Further, for any given constants C and D such that β^2 exhibits the above property, it follows from the fact that V/x is a decreasing function of x that the type (iii) solution does not merge into a type (ii) solution on crossing $\tilde{x}$. For this reason, we require the constants C and D to be so chosen that the type (iii) solution prevails in (x_1, x_0).

4. The determination of C and D

In the following, we shall consider for simplicity $V(x) = 1 - x/x_0$. The condition $\beta^2 > 0$ for x in (x_1, x_0) then yields the relation

$$\tfrac{1}{4}C^2 V/x - \tfrac{1}{2}D < x/V.$$

Since at $x = x_1$ the term on the left attains its maximum value while x/V attains its minimum, the inequality will hold for $x_1 < x < x_0$ if

$$\tfrac{1}{4}C^2 V(x_1)/x_1 - \tfrac{1}{2}D \leqslant x_1/V(x_1).$$

Further, since we do not expect the type (iii) solution to hold for $x \leqslant x_1$, we let $\beta = 0$ at x_1, so that the equality sign holds in the above expression. Rearranging, we have

$$C^2 = 2x_1[2x_1 x_0 + D(x_0 - x_1)]/(x_0 - x_1).$$

If $x_0 \gg x_1$, then roughly we have

$$C^2 = 2x_1(2x_1 + D). \tag{11}$$

The determination of the constants C and D is perforce somewhat arbitrary. However, in so far as is possible, we should like to achieve internal consistency. We observe that from (1) we have

$$u_{zz}(0, x) = V(1 + V/2x). \tag{12}$$

It seems reasonable that we ask $\bar{u}_{zz}$ to satisfy such a condition also. Since

$$\bar{u}_{zz} = V(\alpha^2 + \beta^2) = V(1 + DV/2x),$$

we choose $D = 1$. It then follows from (11) that C is determined by x_1.

To decide on a suitable value for x_1, we have to consider the structure of the hurricane model; see Fendell (1974). For a typical hurricane whose radial extent is O(500 miles), the radius of the eye is O(20 miles). The eye is surrounded by an eyewall, an annulus approximately 10 miles wide. In the eyewall, both the swirl and the radial inflow decrease rapidly from $O(1)$ at the outer edge to near zero at the eye radius. Since we are interested only in that part of the hurricane where

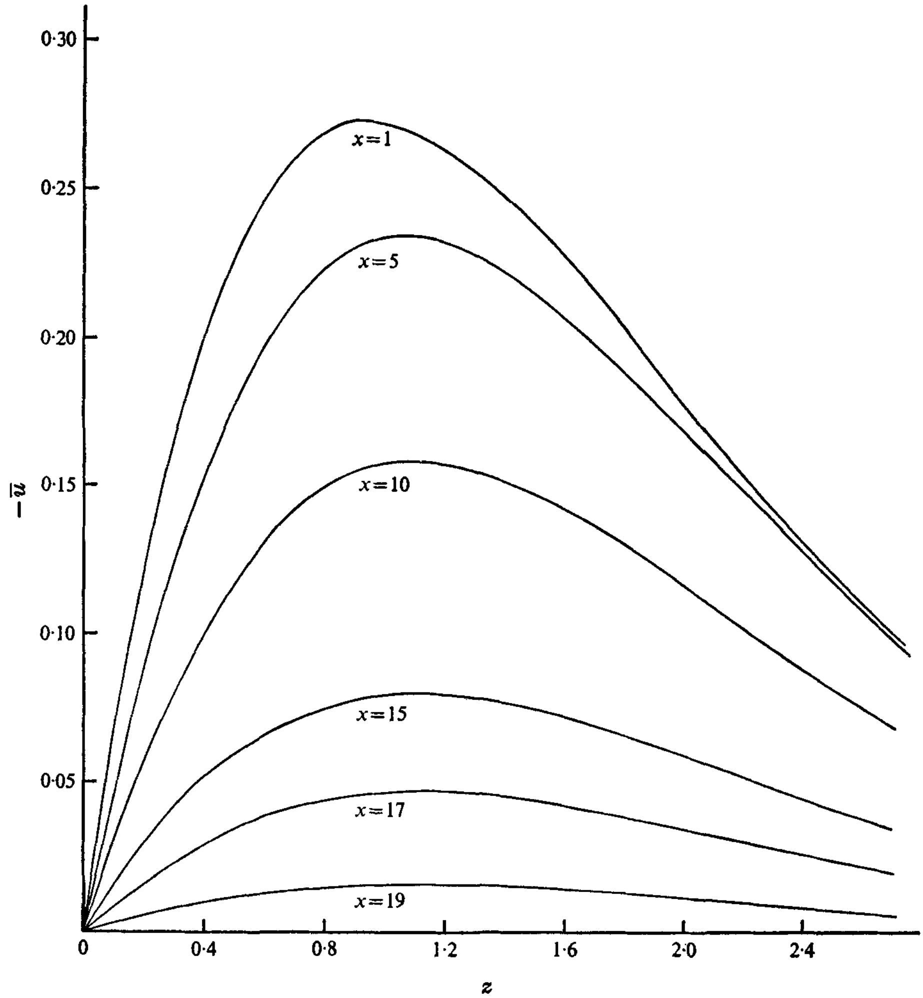

FIGURE 1. Radial velocity profile $\bar{u}$.

$V_r < 0$, x_1 should be no smaller than the outer radius of the eyewall. Guided by the work of Carrier *et al.* (1971), we choose

$$x_0 = 20, \quad x_1 = 0{\cdot}4,$$

which gives

$$C = 1{\cdot}2.$$

If x_0 corresponds to 500 miles, then x_1 corresponds to 70 miles, which is a safe distance outside the eyewall.

With $C = 1{\cdot}2$ and $D = 1$, the functions $\bar{u}$ and $\bar{v}$ are completely determined. A vertical velocity $\bar{w}$ can be calculated from $\bar{u}$ according to the continuity equation. These results are presented in figures 1–3. The question we now pose is how good are $\bar{u}$, $\bar{v}$ and $\bar{w}$ as approximations to the solution of the boundary-value problem (1)–(4)? We observe that the linearized solutions $\bar{u}$ and $\bar{v}$ satisfy all the

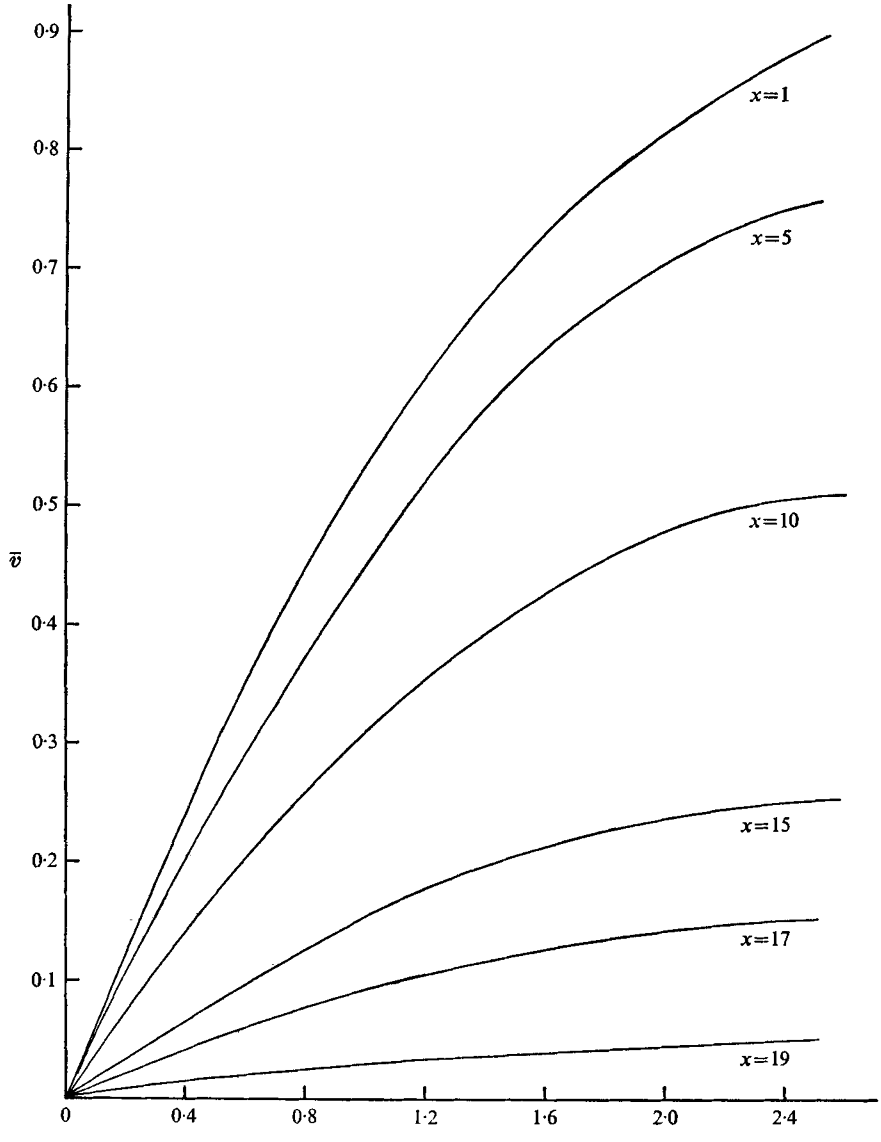

FIGURE 2. Circumferential velocity profile $\bar{v}$.

boundary conditions imposed on the solution of (1)–(4). Thus, in a vicinity of the boundary, $\bar{u}$ and $\bar{v}$ necessarily approximate the true solution.

Now, questions such as the above can be asked of all linearization or approximation schemes. A completely rigorous answer, if one can be found, in general leads to mathematical problems that are just as intractable as the solution of the original nonlinear boundary-value problem. Thus it is not unexpected that only a partial answer can be achieved.

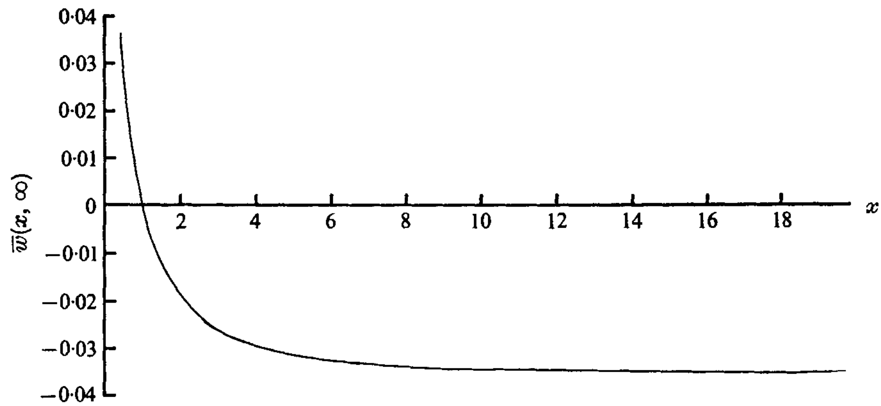

FIGURE 3. Vertical velocity $\bar{w}(x, \infty)$ at the top of the boundary layer.

5. The Nagumo lemma

There exist in the literature a number of comparison theorems for differential equations. For parabolic equations, the central result is due to Nagumo and Westphal (see Walter 1970, p. 187). This result has been used by a number of authors to study the qualitative properties of fluid-mechanical problems, and references to them can be found in the review article by Nickel (1973). Equations (1) and (2) are not parabolic because of the dependence of w on u and v, and so the comparison theorem is not readily applicable. While it is possible to cast (1) and (2) into a parabolic system by using the von Mises transformation

$$\psi(x,z) = \int_0^z u(x,s)\,ds$$

and treat u and v as functions of (x, ψ) instead of (x, z), the domain in the x, ψ plane becomes indeterminate as we have no *a priori* knowledge of the extent in ψ. Thus the gain in simplifying the system is offset.

We observe that the proof of the Nagumo lemma is itself based on a lemma comparing two functions, which is independent of the type of equation they satisfy. It is this basic lemma that we shall make use of. For completeness, we state the lemma and the necessary definitions suited to our purpose. The proof can be found in Walter (1970, p. 185).

Definition.
$$G_p = \{0 < t \leqslant T,\ 0 < z < \bar{z}(t)\},$$
$$R_p = \{t = 0,\ z = 0,\ z = \bar{z}(t)\},$$
$$\bar{G}_p = G_p + R_p.$$

Definition. The function $\phi(t, z)$ belongs to the class Z_0 if it is defined and continuous in G_p and has continuous derivatives ϕ_t, ϕ_z and ϕ_{zz} in G_p.

Lemma. Consider two functions ϕ and $\psi \in Z_0$ and suppose that if $\phi = \psi$, $\phi_z = \psi_z$ and $\phi_{zz} < \psi_{zz}$ at a point in G_p then $\phi_t < \psi_t$ at this point; then we have precisely one of the following cases:

(α) $\phi < \psi$ in G_p.

(β) There exists a maximal $\bar{t}$ $(0 < \bar{t} < T)$ such that $\phi < \psi$ for all points in G_p with $t < \bar{t}$; thus there is a sequence of points $(t_k, z_{(k)}) \in G_p$, $t_k > \bar{t}$, with

$$\phi(t_k, z_{(k)}) \geqslant \psi(t_k, z_{(k)}), \quad k = 1, 2, \ldots,$$

such that, as $k \to \infty$, $(t_k, z_{(k)}) \to (\bar{t}, \bar{x}) \in R_p$.

The essence of (β) is that the inequality $\phi < \psi$ also holds for $t = \bar{t}$ and that consequently $(t_k, z_{(k)})$ always approach a boundary point.

In using the above lemma to answer our question, we proceed as follows. If we can find functions $\tilde{u}$, u^*, $\tilde{v}$ and v^* such that the solutions u and v of the nonlinear problem (1)–(4) satisfy

$$\tilde{u} < u < u^*, \quad \tilde{v} < v < v^*,$$

then we can assess the goodness of the linearized solutions $\bar{u}$ and $\bar{v}$ by comparing them with $\tilde{u}$, u^*, $\tilde{v}$ and v^*. For obvious reasons, u^* and $\tilde{u}$ are referred to as the upper and lower solutions for u respectively, and similarly for v^* and $\tilde{v}$. Of course, if the upper and lower solutions are not sharp, considerable uncertainty will remain.

6. The construction of upper and lower solutions

To cast the system (1)–(4) into a suitable form, we introduce the change of variables

$$t = 20 - x, \quad \phi(t, z) = -u(x, z).$$

Equations (1), (2) and (3) become

$$\phi\phi_t + \left\{w\phi_z - \frac{V^2 - v^2 - \phi^2}{2(20-t)} - (V - v) - \phi_{zz}\right\} = 0, \tag{13}$$

$$\phi v_t + \{wv_z - \phi - v_{zz}\} = 0, \tag{14}$$

$$\phi_t + w_z = 0, \tag{15}$$

and the region of interest is $G_p = \{0 < t < 19{\cdot}6;\ 0 < z < \infty\}$. In terms of the new variables, (5) and (6) become

$$\bar{\phi}_{zz} = \frac{V(t)}{2(20-t)}\bar{\phi} + \left(1 + \frac{V(t)}{2(20-t)}\right)(\bar{v} - V), \tag{16}$$

$$\bar{v}_{zz} = -\bar{\phi}. \tag{17}$$

Since ϕ is required by the boundary conditions to merge with the Ekman solution as $t \to 0$, it is clear that there exists a region adjacent to $z = 0$, at least in a neighbourhood of $t = 0$, such that $\phi > 0$. Specifically, we have $\phi > 0$ for $0 < t < t_1$, $0 < z < \bar{z}(t)$, where t_1 and $\bar{z}$ are positive quantities, whose values have yet to be found. Further, since $\bar{\phi} > 0$ for $0 < z < \pi/\lambda_2$, $0 < t < 19{\cdot}6$, it is clear that in the subdomain

$$\tilde{G}_p = \{0 < t < t_1,\ 0 < z \leqslant \min(\pi/\lambda_2, \bar{z})\}$$

one can find positive quantities k_1 and k_2 such that

$$\tilde{\phi} = (1 - k_2)\bar{\phi} < \phi < (1 + k_1)\bar{\phi} = \phi^*. \tag{18}$$

Similarly, one can find positive quantities h_1 and h_2 such that

$$\tilde{v} = (1-h_2)\,\bar{v} < v < (1+h_1)\,\bar{v} = v^*. \tag{19}$$

In this subdomain, whose existence we have postulated but whose extent we do not yet know, all functions considered are assumed differentiable as many times as required; hence they are in Z_0. The lemma can therefore be applied to compare ϕ^* and $\tilde{\phi}$ with ϕ, and $\tilde{v}$ and v^* with v. What we propose to do is to use the condition of the lemma to determine the extent of $\tilde{G}_p$. Before proceeding, we note that the upper and lower solutions are sought in terms of multiples of the linear solutions $\bar{\phi}$ and $\bar{v}$. Thus, if we should find $k_1 = k_2 = 0{\cdot}5$, then we can say that approximating ϕ by $\bar{\phi}$ incurs a maximum error of 50 %. Moreover, it is clear that since $\bar{\phi}$ and $\bar{v}$ satisfy the same conditions as ϕ and v at $t = 0$ and $z = 0$, it is expected that h_1, h_2, k_1 and k_2 would be small in that vicinity.

If we now apply the lemma to compare ϕ^* and $\tilde{\phi}$ with ϕ, and v^* and $\tilde{v}$ with v, we obtain the four expressions $\phi_t - \phi_t^*$, $\phi_t - \tilde{\phi}_t$, $v_t - v_t^*$ and $v_t - \tilde{v}_t$, which contain the four parameters h_1, h_2, k_1 and k_2. Our objective then is to search for the smallest values of these parameters such that the four inequalities

$$\phi_t - \phi_t^* < 0, \quad \phi_t - \tilde{\phi}_t > 0, \quad v_t - v_t^* < 0, \quad v_t - \tilde{v}_t > 0$$

hold in as large a subdomain as possible. Thus we attempt to use the comparison lemma not merely as a gauging device, but in the actual construction of upper and lower solutions.

Since the quantities $\phi_t - \phi_t^*$, etc. depend explicitly on the vertical velocity w, about which little is known, we have to make some assumptions about w in order to proceed. As observed before, $\bar{\phi}$ is expected to be a good approximation of ϕ in a neighbourhood of $t = 0$. Hence, in that neighbourhood at least, the quantity

$$\bar{w} = \int_0^z \bar{\phi}_t \, dz$$

should approximate w. The expression for $\bar{w}$ is rather unwieldy, and is given in the appendix. We use it as a guide to arrive at the assumption

$$|w| < 0{\cdot}05\,[1 - \exp(-\lambda_1 z)] \equiv \omega^*, \tag{20}$$

which should be a reasonable bound for a wide range of t.

At this point, we must decide whether to let the parameters h_1, h_2, k_1 and k_2 be functions of t and z, or constants. In the first case, the inequalities will lead to inequalities involving the derivatives of the parameters, which would be exceedingly difficult to handle. In the second case, only the parameters themselves enter into consideration. What we can do, of course, is to *assume* that the parameters may depend on t and z but that their derivatives are so small that they can be ignored. We shall adopt this approach in the following computations.

7. The computation of h_1, h_2, k_1 and k_2

We first observe that $\bar{v}_z > 0$ for $\lambda_2 z$ in the range

$$0 < \lambda_2 z < \tfrac{1}{2}\pi + \tfrac{1}{2}\theta, \tag{21}$$

a result readily obtainable from (10). For $V(t) = \frac{1}{20}t$, $\theta = \tan^{-1}\beta/\alpha$ lies between 0 and $\frac{1}{2}\pi$. Hence we have $\bar{\phi} > 0$ for $\lambda_2 z$ in that range also. In the following, we shall restrict z as in (21).

Now, to compare ϕ with ϕ^*, it follows from (13) that, if $\phi = \phi^*$, $\phi_z = \phi_z^*$ and $\phi_{zz} \leqslant \phi_{zz}^*$ at a point in $\tilde{G}_p$, then

$$\phi^*(\phi_t - \phi_t^*) \leqslant -\left\{\phi^*\phi_t^* + w\phi_z^* - \frac{V^2 - v^2 - \phi^{*2}}{2(20-t)} - (V-v)\,\phi_{zz}^*\right\}$$

at that point. Using (16) and (18)–(20), we have

$$\phi^*(\phi_t - \phi_t^*) < -\left\{\phi^*\phi_t^* - \omega^*|\phi_z^*| - \frac{V^2 - \tilde{v}^2 - \phi^{*2}}{2(20-t)} - (V-\tilde{v}) - \phi_{zz}^*\right\},$$

and hence

$$\begin{aligned}\phi^*(\phi_t^* - \phi_t) > {} & (1+k_1)^2\bar{\phi}\bar{\phi}_t - 0{\cdot}05[1-\exp(-\lambda_1 z)]\,(1+k_1)\,|\phi_z| - \frac{V^2-(1-h_2)^2\bar{v}^2}{2(20-t)} \\ & - V + (1-h_2)\bar{v} + \frac{(1+k_1)^2\bar{\phi}^2}{2(20-t)} - (1+k_1)\frac{V\bar{\phi}}{2(20-t)} \\ & - (1+k_1)\left(1+\frac{V}{2(20-t)}\right)(\bar{v}-V) \equiv I_1.\end{aligned}$$

Similarly, if $\phi = \tilde{\phi}$, $\phi_z = \tilde{\phi}_z$ and $\phi_{zz} \geqslant \tilde{\phi}_{zz}$ at a point in $\tilde{G}_p$, then as long as $\tilde{\phi} > 0$, that is $1-k_2 > 0$, we have

$$\begin{aligned}\tilde{\phi}(\tilde{\phi}_t - \phi_t) < {} & (1-k_2)^2\bar{\phi}\bar{\phi}_t + 0{\cdot}05[1-\exp(-\lambda_1 z)]\,(1-k_2)\,|\bar{\phi}_z| - \frac{V^2}{2(20-t)} \\ & + \frac{(1+h_1)^2\bar{v}^2}{2(20-t)} - V + (1+h_1)\bar{v} + \frac{(1-k_2)^2\bar{\phi}^2}{2(20-t)} \\ & - (1-k_2)\frac{V\bar{\phi}}{2(20-t)} - (1-k_2)\left(1+\frac{V}{2(20-t)}\right)(\bar{v}-V) \equiv I_2.\end{aligned}$$

Applying the comparison lemma to v and v^*, we have that, if $v = v^*$, $v_z = v_z^*$ and $v_{zz} \leqslant v_{zz}^*$ at a point in $\tilde{G}_p$, then

$$\bar{\phi}(v_t^* - v_t) > (1+h_1)\,\bar{\phi}\bar{v}_t - \frac{h_1-k_1}{1+k_1}\bar{\phi} - \frac{1}{1-k_2}\,0{\cdot}05(1+h_1)\,[1-\exp(-\lambda_1 z)]\,\bar{v}_z \equiv I_3.$$

Finally, if $v = v$, $\tilde{v}_z = \tilde{v}_z$ and $v_{zz} \geqslant \tilde{v}_{zz}$ at a point in $\tilde{G}_p$, then

$$\begin{aligned}\tilde{\phi}(\tilde{v}_t - v_t) < {} & (1-h_2)\,(1-k_2)\,\bar{\phi}\bar{v}_t + (k_2-h_2)\,\bar{\phi} \\ & + 0{\cdot}05[1-\exp(-\lambda_1 z)]\,(1-h_2)\bar{v}_z \equiv I_4.\end{aligned}$$

To recapitulate, our objective is to search for the smallest parameters h_1, h_2, k_1 and k_2 such that $I_1 \geqslant 0$, $I_2 \leqslant 0$, $I_3 \geqslant 0$ and $I_4 \leqslant 0$ in as large a subdomain of the region

$$R = \{0 < t < 19{\cdot}6,\ 0 < z < (2\lambda_2)^{-1}(\pi+\theta)\}$$

as possible.

We cover R by a grid. The mesh size in z is 0·1, and the mesh size in t is 2 for $t = 2$–18 and 0·1 otherwise. Numerical computations are performed at the grid points. The parameters h_1 and h_2 are assigned the values 0·1, 0·2, ..., 0·5 and the parameters k_1 and k_2 are assigned the values 0·1, 0·2, ..., 0·7. Each combination

z	$t = 2$	$t = 4$	$t = 6$	$t = 8$
0·1	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)
0·2	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)
0·3	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)
0·4	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)
0·5	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2. 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)
0·6	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)
0·7	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)
0·8	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)
0·9	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)
1·0	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)
1·1	(0·1, 0·1, 0·2, 0·3)	(0·1, 0·1, 0·2, 0·3)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·3, 0·2)
1·2	(0·1, 0·1, 0·3, 0·3)	(0·1, 0·1, 0·3, 0·3)	(0·1, 0·1, 0·3, 0·3)	(0·1, 0·1, 0·3, 0·3)
1·3	(0·1, 0·1, 0·3, 0·4)	(0·1, 0·1, 0·3, 0·4)	(0·1, 0·1, 0·3, 0·4)	(0·1, 0·1, 0·4, 0·3)
1·4	(0·1, 0·1, 0·4, 0·4)	(0·1, 0·1, 0·4, 0·4)	(0·1, 0·1, 0·4, 0·4)	(0·1, 0·1, 0·4, 0·4)

z	$t = 10$	$t = 12$	$t = 14$	$t = 16$
0·1	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)
0·2	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)
0·3	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)
0·4	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)
0·5	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)
0·6	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)
0·7	(0·1, 0·1, 0·2, 0·2)	(0·1. 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)
0·8	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)
0·9	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·2, 0·2, 0·2)	(0·1, 0·1, 0·3, 0·2)
1·0	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·3, 0·2)	(0·1, 0·1, 0·3, 0·2)
1·1	(0·1, 0·1, 0·3, 0·2)	(0·1, 0·1, 0·3, 0·2)	(0·1, 0·1, 0·3, 0·2)	(0·1, 0·1, 0·4, 0·2)
1·2	(0·1, 0·1, 0·3, 0·3)	(0·1, 0·1, 0·3, 0·3)	(0·1, 0·1, 0·4, 0·2)	(0·1, 0·1, 0·4, 0·2)
1·3	(0·1, 0·1, 0·4, 0·3)	(0·1, 0·1, 0·4, 0·3)	(0·1, 0·1, 0·4, 0·3)	
1·4	(0·1, 0·1, 0·4, 0·4)			

z	$t = 18$	$t = 18·3$	$t = 18·6$	$t = 18·9$
0·1	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)
0·2	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)
0·3	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)
0·4	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)
0·5	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)
0·6	(0·1, 0·1, 0·2, 0·3)	(0·1, 0·1, 0·3, 0·2)	(0·1, 0·1, 0·2, 0·3)	(0·1, 0·1, 0·2, 0·2)
0·7	(0·1, 0·1, 0·3, 0·2)	(0·1, 0·1, 0·3, 0·2)	(0·1, 0·1, 0·3, 0·2)	(0·1, 0·1, 0·2, 0·2)
0·8	(0·1, 0·1, 0·3, 0·2)	(0·1, 0·1, 0·3, 0·2)	(0·1, 0·1, 0·3, 0·2)	(0·1, 0·1, 0·2, 0·2)
0·9	(0·1, 0·1, 0·4, 0·2)	(0·1, 0·1, 0·4, 0·2)	(0·1, 0·1, 0·3, 0·3)	(0·1, 0·1, 0·2, 0·3)
1·0	(0·1, 0·1, 0·4, 0·2)	(0·1, 0·1, 0·4, 0·2)	(0·1, 0·1, 0·4, 0·2)	(0·1, 0·1, 0·3, 0·2)
1·1			(0·1, 0·1, 0·4, 0·2)	(0·1, 0·1, 0·3, 0·2)
1·2				(0·1, 0·1, 0·3, 0·2)
1·3				(0·1, 0·1, 0·3, 0·4)
1·4				(0·1, 0·1, 0·4, 0·2)
1·5				(0·1, 0·1, 0·4, 0·3)

TABLE 1. Tabulation of the 4-tuple $(\bar{h}_1, \bar{h}_2, \bar{k}_1, \bar{k}_2)$ in $\tilde{G}_p$

z	$t = 19{\cdot}1$	$t = 19{\cdot}2$	$t = 19{\cdot}3$	$t = 19{\cdot}4$
0·1	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)
0·2	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)
0·3	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)
0·4	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·3, 0·3)
0·5	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·3, 0·3)
0·6	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·3, 0·4)
0·7	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·3, 0·3)	(0·1, 0·1, 0·3, 0·4)
0·8	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·3, 0·3)	(0·1, 0·1, 0·3, 0·4)
0·9	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·3, 0·3)	(0·1, 0·1, 0·3, 0·4)
1·0	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·3, 0·2)	(0·1, 0·1, 0·3, 0·3)	
1·1	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·3, 0·2)	(0·1, 0·1, 0·3, 0·3)	
1·2	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·3, 0·2)	(0·1, 0·1, 0·3, 0·3)	
1·3	(0·1, 0·1, 0·2, 0·2)	(0·1, 0·1, 0·3, 0·2)	(0·1, 0·1, 0·3, 0·4)	
1·4	(0·1, 0·1, 0·3, 0·2)	(0·1, 0·1, 0·3, 0·3)	(0·1, 0·1, 0·3, 0·4)	
1·5	(0·1, 0·1, 0·3, 0·2)	(0·1, 0·1, 0·3, 0·3)	(0·1, 0·1, 0·3, 0·4)	
1·6	(0·1, 0·1, 0·3, 0·2)	(0·1, 0·1, 0·3, 0·3)	(0·1, 0·1, 0·3, 0·4)	
1·7	(0·1, 0·1, 0·3, 0·3)	(0·1, 0·1, 0·3, 0·3)	(0·1, 0·1, 0·3, 0·4)	
1·8	(0·1, 0·1, 0·3, 0·4)	(0·1, 0·1, 0·3, 0·4)	(0·1, 0·1, 0·3, 0·4)	
1·9	(0·1, 0·1, 0·4, 0·4)	(0·1, 0·1, 0·3, 0·4)		

TABLE 1 (*cont.*)

of h_1, h_2, k_1 and k_2 that satisfies the four inequalities simultaneously is printed out. As expected, at a given $t < 19{\cdot}6$, the number of admissible combinations is large for $z < 1$, and h_1, h_2, k_1 and k_2 take on relatively small values. As z increases, the number of admissible combinations decreases rapidly, with k_1 and k_2 taking larger values. For $z > 1{\cdot}5$, virtually no combination within the assigned parameter range is possible. In table 1, we list the combinations with the smallest value for each parameter at each grid point, for a few representative values of t. Those grid points (for a fixed t) with any parameter in the combination larger than 0·4 are left out.

We conclude from the numerical results that in the subdomain

$$\tilde{G}_p = \{0 < t < 19{\cdot}4,\ 0 < z < \bar{z}\},$$

where $\bar{z}$ is the largest value of z listed in table 1, the conditions required by the lemma for the comparison of ϕ with ϕ^* and $\tilde{\phi}$, and v with v^* and $\tilde{v}$ are satisfied. Further, for any given arbitrarily small $\epsilon > 0$, if we now set

$$\left.\begin{aligned} \phi^* &= (1+\bar{k}_1)\bar{\phi}+\epsilon, & \tilde{\phi} &= (1-\bar{k}_2)\bar{\phi}-\epsilon,\\ v^* &= (1+\bar{h}_1)\bar{v}+\epsilon, & \tilde{v} &= (1-\bar{h}_2)\bar{v}-\epsilon, \end{aligned}\right\} \tag{22}$$

the inequalities $I_1 > 0$, $I_2 < 0$, $I_3 > 0$ and $I_4 < 0$ still hold. With this modification, it is clear that

$$\tilde{\phi} < \phi < \phi^*, \quad \tilde{v} < v < v^*$$

at $t = 0$ and $z = 0$. By our construction, we have

$$\tilde{\phi}(t,\bar{z}) < \phi(t,\bar{z}) < \phi^*(t,\bar{z}),$$

$$\tilde{v}(t,\bar{z}) < v(t,\bar{z}) < v^*(t,\bar{z}),$$

thus alternative (β) of the lemma cannot hold. We summarize our results in the following statement.

Statement. Under the assumption that $|w| < 0{\cdot}05(1-\exp(-\lambda_1 z)]$, the solutions ϕ and v of the boundary-value problem (1)–(4) satisfy the inequalities

$$\left.\begin{aligned}(1-\bar{k}_2)\bar{\phi}-\epsilon < \phi < (1+\bar{k}_1)\bar{\phi}+\epsilon\\ (1-\bar{h}_2)\bar{v}-\epsilon < v < (1+\bar{h}_1)\bar{v}+\epsilon\end{aligned}\right\} \quad \text{in} \quad \tilde{G}_p,$$

where ϵ is arbitrarily small, $\bar{\phi}$ and $\bar{v}$ are as given in (9) and (10), and $\bar{h}_1$, $\bar{h}_2$, $\bar{k}_1$, $\bar{k}_2$ and $\tilde{G}_p$ are as described in table 1.

8. Concluding remarks

(1) It is clear that, while $\tilde{G}_p$ does not extend very far in the vertical direction, its thickness is of the order of the e-folding thickness $1/\lambda_1$ of the boundary layer. Also, the inequalities used in the computation of h_1, etc., are obtained by making some 'overpowering' majorizations. It is thus not unreasonable to expect that, within the percentage error range given by $\bar{h}_1$, etc., $\tilde{G}_p$ may extend further in the z direction.

(ii) Since $\bar{\phi}$ decays exponentially in the z direction, and ϕ is expected to have the same behaviour, the downdraft (or updraft)

$$w(x,\infty) = -\int_0^\infty \phi_t\,dz$$

receives its major contribution from

$$-\int_0^a \phi_t\,dz,$$

where $a = O(1)$. From the bounds constructed, we *infer* that

$$\bar{w}(x,\infty) = -\int_0^\infty \bar{\phi}_t\,dz$$

approximates $w(x,\infty)$ to the same degree of accuracy as $\bar{\phi}$ approximates ϕ, which is about $\pm 30\,\%$ in $0 < t < 19{\cdot}4$.

(iii) Since the determination of C is not as clear cut as that for D, we have performed similar, though less extensive, calculations for neighbouring values of C. No drastic departure occurs, but the smaller the value of C, the closer to the eye is the reversal of the downdraft into a stronger updraft. For $0{\cdot}866 < C < 1{\cdot}414$, the error bounds are more or less the same as reported. Since a smaller C implies a smaller x_1, and hence a smaller eye, this trend seems consistent with the observation that "the more intense storm might generally have a relatively small eye" (Fendell 1974).

(iv) We have used a comparison theorem for the purpose of constructing upper and lower solutions. We believe this procedure can be exploited in similar problems, and indeed have used it to consider a few other examples.

This work was initiated in 1972 while the author was visiting Harvard University. The general conclusion reached in this paper has been cited by

Fendell (1974). The author thanks Professor George Carrier for suggesting the problem, and for many stimulating discussions. Thanks are also due to Mr P. Chu for doing the computations. Continuing support from the National Research Council of Canada is acknowledged.

Appendix

We list here a number of quantities obtained from $\bar{\phi}$ and $\bar{v}$ which are required in the computation of $I_1, \ldots, I_4$.

$$\begin{aligned}\bar{w}(x,z) = {} & \bar{w}(x,\infty)\,[1-\exp(-\lambda_1 z)\cos\lambda_2 z] \\ & +\frac{1}{(\alpha^2+\beta^2)^{\frac{1}{2}}}\left[\bar{u}\frac{d\lambda_1}{dx}+\lambda_1\bar{u}_x-\frac{\lambda_1\bar{u}}{2}\frac{d}{dx}(\alpha^2+\beta^2)\right] \\ & +\frac{\lambda_2 V(\alpha^2+\beta^2)^{\frac{1}{2}}}{\beta}\exp(-\lambda_1 z)\left[z\cos\lambda_2 z\frac{d\lambda_1}{dx}+z\sin\lambda_2 z\frac{d\lambda_2}{dx}\right],\end{aligned}$$

where
$$\bar{w}(x,\infty) = \frac{d}{dx}\left[\frac{\lambda_2 V(\alpha^2+\beta^2)^{\frac{1}{2}}}{\beta}\right],$$

$$\begin{aligned}\bar{\phi}_t = {} & \frac{d}{dt}\left[\frac{V(\alpha^2+\beta^2)}{\beta}\right]\exp(-\lambda_1 z)\sin\lambda_2 z+\frac{V(\alpha^2+\beta^2)}{\beta}z\exp(-\lambda_1 z) \\ & \times\left(\cos\lambda_2 z\frac{d\lambda_2}{dt}-\sin\lambda_2 z\frac{d\lambda_1}{dt}\right),\end{aligned}$$

$$\bar{\phi}_z = V(\alpha^2+\beta^2)^{\frac{3}{4}}\,\beta^{-1}\exp(-\lambda_1 z)\sin(\tfrac{1}{2}\theta-\lambda_2 z),$$

$$\begin{aligned}\bar{v}_t = {} & V_t[1-\exp(-\lambda_1 z)\cos\lambda_2 z]+\exp(-\lambda_1 z)\sin\lambda_2 z\left(-V_t\cot\theta+\frac{V}{\sin^2\theta}\frac{d\theta}{dt}\right) \\ & +z\exp(-\lambda_1 z)\sin\lambda_2 z\left[V\cot\theta\frac{d\lambda_1}{dt}+V\frac{d\lambda_2}{dt}\right] \\ & +z\exp(-\lambda_1 z)\cos\lambda_2 z\left[V\frac{d\lambda_1}{dt}-V\cot\theta\frac{d\lambda_2}{dt}\right],\end{aligned}$$

$$\bar{v}_z = \frac{V\exp(-\lambda_1 z)}{\sin\theta}(\lambda_1\sin\lambda_2 z\cos\lambda_2 z).$$

The values of α, β, θ, λ_1 and λ_2 are given in § 3.

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